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The polynomial spectral problem of arbitrary order: A general form of the integrable equations and Bäcklund transformations

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Abstract. The generalisation of the AKNS method to the matrix polynomial spectral problem of arbitrary order is given. Both the general form of the integrable partial differential equations and their Bäcklund transformations are described.

1. Introduction

The inverse spectral transform (IST) method allows a comprehensive study of a great number of various partial differential equations (see e.g. Scott *et al* 1973, Bullough and Caudrey 1980, Zakharov *et al* 1980). One of the main problems of the IST method is that of the description of the differential equations which are integrable by this method.

All the partial differential equations to which the IST method is applicable are grouped into classes of equations which are integrable by the same linear spectral problem. A simple and convenient description of the class of equations integrable by the linear (in spectral parameter) spectral problem of the second order was presented by AKNS (Ablowitz *et al* 1974). This class of equations is characterised by the $(m-1)$ arbitrary functions (m is the number of independent variables) and by a certain integro-differential operator (Ablowitz *et al* 1974, Calogero and Degasperis 1976). The analogous result was obtained for the class of equations which are associated with the matrix stationary Schrödinger equation (Calogero and Degasperis 1977). The AKNS method has been extended to the general matrix linear spectral problem of arbitrary order (Newell 1978, 1979, Miodek 1978, Kulish 1979, Konopelchenko 1980a, b, c). The second-order linear spectral problem, quadratic in spectral parameter, was considered by Gerdjikov *et al* (1980). Within the framework of this approach the wide classes of Bäcklund transformations (BT's) which play a significant role in the study of nonlinear differential equations are also found (Calogero and Degasperis 1976, 1977, Dodd and Bullough 1977, Konopelchenko 1980a, c, d, Gerdjikov *et al* 1980).

In the present paper we generalise the AKNS method to the general polynomial matrix spectral problem of arbitrary order

$$\frac{\partial\psi}{\partial x} = P\psi = \sum_{\alpha=0}^{n} \lambda^{\alpha} P^{(\alpha)}(x, t)\psi \tag{1.1}$$

where λ is the spectral parameter, $P^{(\alpha)}(x, t)$ are matrices of order N, n is an arbitrary number, and x and t are independent variables (coordinate and time). The applicability of the IST method to polynomial spectral problems was discussed by Zakharov (1980).

We describe a general form of the differential equations integrable by (1.1) and the general form of the Bäcklund transformations for these equations. The universal nonlinear transformation group, which contains the group of Bäcklund transformations and the integrable equations themselves, is constructed. Infinite series of the integrals of motion are calculated. As an example the linear ($n=1$) and quadratic ($n=2$) bundles (1.1) of arbitrary order N are considered. The infinite class of the integrable equations connected to the quadratic bundle contains multicomponent and matrix generalisations of the derivative and combined nonlinear Schrödinger equations. The equivalence of the polynomial spectral problem (1.1), as well as the general spectral problem, rational in the spectral parameter λ, to the degenerate spectral problem, linear in λ, is also pointed out.

The paper is arranged as follows. In § 2 the infinite-dimensional group of nonlinear transformations connected with the spectral problem (1.1) is constructed. In § 3 it is shown that this group contains the integrable equations and Bäcklund transformations. Two examples—linear and quadratic bundles—are considered in §§ 4 and 5. The equivalence of (1.1) to the degenerate spectral problem, linear in λ, is discussed in § 6.

2. Nonlinear transformations connected with problem (1.1)

2.1. *The transition matrix S and its transformation*

We assume that

$$P^{(\alpha)}(x,t)\xrightarrow[|x|\to\infty]{}A^{(\alpha)}\qquad(\alpha=1,\ldots,n),$$

where $A^{(\alpha)}$ are constant matrices, which commute with each other; $[A^{(\alpha)},A^{(\beta)}]=0$, $(\alpha,\beta=1,\ldots,n)$. In other words we assume that all $A^{(\alpha)}$ $(\alpha=1,\ldots,n)$ belong to a certain Cartan subalgebra $g_{0(A)}$ of the full matrix algebra $gl(N,C)$. For properties of Cartan subalgebras see Bourbaki (1972). In particular, all Cartan subalgebras of $gl(N,C)$ are commutative and they have the dimension N. Let us denote a basis for the Cartan subalgebra $g_{0(A)}$ as $H_{(A)i}$ $(i=1,\ldots,N)$. Then there exists a decomposition of $gl(N,C)$ into the direct sum $gl(N,C)=g_{0(A)}\oplus g_{F(A)}$, where $g_{0(A)}=\{g, g\in gl(N,C),[g,A]=0\}$. For an arbitrary matrix Φ of order N we obtain a decomposition $\Phi=\Phi_{0(A)}+\Phi_{F(A)}$ where $\Phi_{0(A)}$ is a projection of Φ onto $g_{0(A)}$ and $\Phi_{F(A)}$ is a projection of Φ onto $g_{F(A)}$. In particular $P^{(\alpha)}=P^{(\alpha)}_{0(A)}+P^{(\alpha)}_{F(A)}$. So our assumption means that

$$P^{(\alpha)}_{0(A)}(x,t)\xrightarrow[|x|\to\infty]{}A^{(\alpha)},\qquad P^{(\alpha)}_{F(A)}(x,t)\xrightarrow[|x|\to\infty]{}0,$$

i.e.

$$P(x,t)\xrightarrow[|x|\to\infty]{}\sum_{\alpha=0}^{n}\lambda^{\alpha}A^{(\alpha)}\in g_{0(A)}.$$

For simplicity we will also assume that $P^{(\alpha)}_{0(A)}\equiv A^{(\alpha)}$ $(\alpha=1,\ldots,n)$.

By virtue of this assumption for solution $\psi(x, t, \lambda)$ of (1.1) we have:

$$\psi(x, t, \lambda) \xrightarrow[|x|\to\infty]{} E = \exp\left(\sum_{\alpha=0}^{n} \lambda^{\alpha} A^{(\alpha)} x\right) \in g_{0(A)}.$$

Let us introduce in the usual way (see e.g. Zakharov *et al* 1980) the fundamental matrix solutions F^+ and F^- with asymptotics

$$F^+ \xrightarrow[x\to+\infty]{} E, \qquad F^- \xrightarrow[x\to-\infty]{} E$$

and the transition matrix

$$S(\lambda, t): F^+(x, t, \lambda) = F^-(x, t, \lambda) S(\lambda, t).$$

The system of the linear equations (1.1) gives a mapping $P(x, t) \to \psi(x, t, \lambda) \to S(\lambda, t)$. Let P and P' be two 'potentials' in (1.1) and ψ and ψ' two corresponding solutions of (1.1). It isn't difficult to show that (for $N = 2$ see Dodd and Bullough, 1977)

$$\psi' - \psi = -\psi \int_x^{\infty} \mathrm{d}y\, \psi^{-1}(P' - P)\psi'.$$

Putting $\psi = F^+$ and proceeding to the limit $x \to -\infty$ we get

$$S' - S = -S \int_{-\infty}^{+\infty} \mathrm{d}x (F^+)^{-1}(P' - P)(F^+)'. \tag{2.1}$$

The formula (2.1) connecting the variation of the 'potentials' $P^{(\alpha)}$ with the variation of the transition matrix S is a basis for the further discussion†.

The mapping $P^{(\alpha)}(x, t) \to S(\lambda, t)$ given by (1.1) exhibits a correspondence between the transformations $P^{(\alpha)} \to P'^{(\alpha)}$ on the manifold of the 'potentials' $\{P^{(\alpha)}(x, t)\}$ and $S(\lambda, t) \to S'(\lambda, t)$ on the set of transition matrices $\{S(\lambda, t)\}$ which is given by

$$\begin{array}{ccc} P & \xrightarrow{(1.1)} & S \\ {\scriptstyle \mathbf{T}_P}\downarrow & & \downarrow{\scriptstyle \mathbf{T}_S} \\ P' & \xrightarrow{(1.1)} & S'. \end{array} \tag{2.2}$$

Let us now consider only those transformations $\mathbf{T}$ for which

$$\mathrm{S}(\lambda, \mathrm{t}) \xrightarrow{\mathbf{T}} S'(\lambda, t) = B^{-1}(\lambda, t) S(\lambda, t) C(\lambda, t) \tag{2.3}$$

where $B(\lambda, t) = B_{0(A)}(\lambda, t) \in g_{0(A)}$ and $C(\lambda, t) = C_{0(A)}(\lambda, t) \in g_{0(A)}$. So we are restricted by those transformations $P \to P'$ for which the transition matrix S transforms in a simple linear manner (2.3). The assumption (2.3) is the generalisation of the main idea behind the IST method which consists in the mapping of the nonlinear evolution law of the potential P due to the nonlinear differential equation onto the linear evolution law (easily integrable) of the transition matrix S.

The transformations (2.3) are defined by matrices $B(\lambda, t)$ and $C(\lambda, t)$. The main advantage of the generalised AKNS method consists of the construction in the explicit

† For the linear (in λ) spectral problem see Konopelchenko (1980a–c) and, in the infinitesimal form, Shabat (1979).

form of the transformation $P \to P'$ which corresponds to the transformation $S \to S'$ of type (2.3) (for $N=2$, $n=1$, $P^{(1)}_{F(A)}=0$, $A^{(0)}=0$, and $A^{(1)}$ is a diagonal matrix (see Dodd and Bullough 1977, 1979); for $n=1$, $P^{(1)}_{F(A)}=0$, $A^{(0)}=0$, arbitrary semisimple $A^{(1)}$ and arbitrary N (see Konopelchenko, 1980a, b, c).

Let us do this for the general polynomial bundle (1.1). Rewriting (2.3) in the form $S'-S=(\mathbb{1}-B)S'-S(1-C)$ and comparing with (2.1) we get

$$(S^{-1}(\mathbb{1}-B)S')_{F(A)}=-\int_{-\infty}^{+\infty}\mathrm{d}x((F^+)^{-1}(P'-P)(F^+)')_{F(A)}. \tag{2.4}$$

Then by virtue of the relations $F^+\xrightarrow[x\to+\infty]{}E$, $(F^+)'\xrightarrow[x\to+\infty]{}E$ (we assume $\lim_{|x|\to\infty}P^{(\alpha)}=\lim_{|x|\to\infty}P'^{(\alpha)}=A^{(\alpha)}$), $[B,E]=0$ and $E_{F(A)}=0$, the following equality holds

$$\begin{aligned}(S^{-1}(\mathbb{1}-B)S')_{F(A)}&\\ &=-\int_{-\infty}^{+\infty}\mathrm{d}x\frac{\partial}{\partial x}((F^+)^{-1}(\mathbb{1}-B)(F^+)')_{F(A)}\\ &=\int_{-\infty}^{+\infty}\mathrm{d}x\{(F^+)^{-1}[P(\mathbb{1}-B)-(\mathbb{1}-B)P'](F^+)'\}_{F(A)}.\end{aligned} \tag{2.5}$$

Combining (2.4) and (2.5) we get

$$\int_{-\infty}^{+\infty}\mathrm{d}x((F^+)^{-1}(BP'-PB)(F^+)')_{F(A)}=0. \tag{2.6}$$

Rewriting equation (2.6) in components and designating

$$(\overset{++}{\phi}{}^{(ip)})_{kl}\overset{\mathrm{df}}{=}(F^+)'_{kn}(F^+)^{-1}_{il}\qquad (i,k,n,l=1,\ldots,N)$$

we obtain

$$\langle(BP'-PB),\overset{++}{\phi}{}^{F(A)}\rangle=0 \tag{2.7}$$

where $\langle\chi,\psi\rangle\overset{\mathrm{df}}{=}\int_{-\infty}^{+\infty}\mathrm{d}x\,\mathrm{Tr}(\chi(x)\psi(x))$ and Tr denotes the usual matrix trace. Since $B(\lambda,t)=\sum_{i=1}^{N}B_i(\lambda,t)H_{(A)i}$ we have

$$\left\langle\sum_{i=1}^{N}(H_{(A)i}P'-PH_{(A)i}),B_i(\lambda,t)\overset{++}{\phi}{}^{F(A)}\right\rangle=0. \tag{2.8}$$

Then due to the relations $H_{(A)i}P'-PH_{(A)i}=H_{(A)i}P'_{F(A)}-P_{F(A)}H_{(A)i}$ and $\mathrm{Tr}(\psi_{F(A)}\phi)=\mathrm{Tr}(\psi_{F(A)}\phi_{F(A)})$ the equation (2.8) is equivalent to the following

$$\begin{aligned}&\left\langle\sum_{i=1}^{N}(H_{(A)i}P'-PH_{(A)i}),B_i(\lambda,t)\overset{++}{\phi}{}^{F(A)}_{F(A)}\right\rangle\\ &\qquad=\sum_{i=1}^{N}\sum_{\alpha=0}^{n}\langle(H_{(A)i}P'^{(\alpha)}-P^{\alpha}H_{(A)i}),\lambda^{\alpha}B_i(\lambda,t)\overset{++}{\phi}{}^{F(A)}_{F(A)}\rangle=0.\end{aligned} \tag{2.9}$$

2.2. *The Λ operators*

The relation (2.9) contains the product $\lambda^{\alpha}B_i(\lambda,t)\overset{++}{\phi}{}^{F(A)}_{F(A)}(x,t,\lambda)$ which is given locally (in each point λ of the bundle (1.1)). The spectral problem (1.1) allows the possibility of transforming this local (in λ) product into a global one which is determined for the whole bundle.

To do this let us obtain the equation for $\overset{++}{\phi}(x, t, \lambda)$. From the definition of $\overset{++}{\phi} \overset{\mathrm{df}}{=} (F^+)' \otimes (F^+)^{-1}$ and (1.1) we have

$$\frac{\partial \overset{++}{\phi}{}^{(ip)}}{\partial x} = \left[\sum_{\alpha=0}^{n} \lambda^\alpha A^{(\alpha)}, \overset{++}{\phi}{}^{(ip)} \right] + P'_{F(A)} \overset{++}{\phi}{}^{(ip)} - \overset{++}{\phi}{}^{(ip)} P_{F(A)} \qquad (i, p = 1, \ldots, N). \tag{2.10}$$

From (2.10) it follows that

$$\partial \overset{++}{\phi}{}^{(ip)}_{0(A)} / \partial x = (P'_{F(A)} \overset{++}{\phi}{}^{(ip)}_{F(A)} - \overset{++}{\phi}{}^{(ip)}_{F(A)} P_{F(A)})_{0(A)}. \tag{2.11}$$

As a result

$$(\overset{++}{\phi}{}^{(ip)}_{(x,t,\lambda)})_{0(A)} = (\overset{++}{\phi}{}^{(ip)}_{(x=+\infty)})_{0(A)} - \int_x^\infty \mathrm{d}y (P'_{F(A)}(y) \overset{++}{\phi}{}^{(ip)}_{F(A)}(y)$$
$$- \overset{++}{\phi}{}^{(ip)}_{F(A)}(y) P_{F(A)}(y))_{0(A)} \qquad (i, p = 1, \ldots, N). \tag{2.12}$$

Since $F^+ \xrightarrow[x \to +\infty]{} E = \exp(\Sigma_{\alpha=0}^n \lambda^\alpha A^{(\alpha)} x)$ we have $(\overset{++}{\phi}{}^{F(A)}_{(x=+\infty)})_{0(A)} = 0$. Then substituting (2.12) into (2.10) and taking the $F(A)$ component of (2.10) gives

$$\frac{\partial \overset{++}{\phi}{}^{F(A)}}{\partial x} = \left[\sum_{\alpha=0}^{n} \lambda^\alpha A^{(\alpha)}, \overset{++}{\phi}{}^{F(A)}_{F(A)}(x) \right]$$
$$+ \sum_{\alpha=0}^{n} \lambda^\alpha (P'^{(\alpha)}_{F(A)}(x) \overset{++}{\phi}{}^{F(A)}_{F(A)}(x) - \overset{++}{\phi}{}^{F(A)}_{F(A)}(x) P^{(\alpha)}_{F(A)}(x))_{F(A)}$$
$$- \sum_{\alpha=0}^{n} \sum_{\beta=0}^{n} \lambda^{\alpha+\beta} P'^{(\alpha)}_{F(A)}(x)$$
$$\times \int_x^\infty \mathrm{d}y (P'^{(\beta)}_{F(A)}(y) \overset{++}{\phi}{}^{F(A)}_{F(A)}(y) - \overset{++}{\phi}{}^{F(A)}_{F(A)}(y) P^{(\beta)}_{F(A)}(y))_{0(A)}$$
$$+ \sum_{\alpha=0}^{n} \sum_{\beta=0}^{n} \lambda^{\alpha+\beta} \int_x^\infty \mathrm{d}y (P'^{(\beta)}_{F(A)}(y) \overset{++}{\phi}{}^{F(A)}_{F(A)}(y)$$
$$- \overset{++}{\phi}{}^{F(A)}_{F(A)}(y) P^{(\beta)}_{F(A)}(y))_{0(A)} P^{(\alpha)}_{F(A)}(x). \tag{2.13}$$

Thus the quantity $\overset{++}{\phi}{}^{F(A)}_{F(A)}$ satisfies the matrix integro-differential equation (2.13) which is a polynomial in λ. Equation (2.13) can be rewritten in compact form

$$\bar{f}(\lambda) \overset{++}{\phi}{}^{F(A)}_{F(A)} \overset{\mathrm{df}}{=} \left(\sum_{\gamma=0}^{2n} \lambda^\gamma Q_{(\gamma)} \right) \overset{++}{\phi}{}^{F(A)}_{F(A)} = 0 \tag{2.14}$$

where $Q_{(\gamma)}$ are matrices with operator elements. The explicit form of $Q_{(\gamma)}$ in the terms $P^{(\alpha)}$ and $P'^{(\alpha)}$ can be easily obtained from equations (2.13) and (2.14).

By means of the generalised Bezout theorem (see e.g. Gantmakher 1967) equation (2.14) can be represented as follows

$$Q(\lambda)\prod_{\rho=1}^{M}(\lambda-\Lambda_{(\rho)})\overset{++}{\phi}{}^{F(A)}_{F(A)}=0 \tag{2.15}$$

where the operators $\Lambda_{(\rho)}$ are 'roots' of the operator polynomial $\bar{f}(\lambda)$:

$$\bar{f}(\Lambda_{(\rho)})=0 \qquad (\rho=1,\ldots,M;\, M\leqslant 2n).$$

The equivalence of right and left division (Gantmakher 1967) leads to the commutativity of the factors $\lambda-\Lambda_{(\rho)}$ in the product $\Pi_\rho\,(\lambda-\Lambda_{(\rho)})$. In view of this property, the solutions of equation (2.15) can be represented in the form (assuming the non-degeneracy of the operator $Q(\lambda)$):

$$\overset{++}{\phi}{}^{F(A)}_{F(A)}=\sum_{\rho=1}^{M}\overset{++}{\phi}{}^{F(A)}_{(\rho)F(A)} \tag{2.16}$$

where

$$(\lambda-\Lambda_{(\rho)})\overset{++}{\phi}{}^{F(A)}_{(\rho)F(A)}=0 \qquad (\rho=1,\ldots,M). \tag{2.17}$$

So $\overset{++}{\phi}{}^{F(A)}_{(\rho)F(A)}$ is the eigenfunction of the operator $\Lambda_{(\rho)}$ and λ is an eigenvalue.

From (2.17) follows that for any entire function $f(\lambda)$

$$f(\lambda)\overset{++}{\phi}{}^{F(A)}_{(\rho)F(A)}=f(\Lambda_{(\rho)})\overset{++}{\phi}{}^{F(A)}_{(\rho)F(A)}. \tag{2.18}$$

2.3. The transformations of the potentials

Let us now return to equation (2.9). In view of (2.16) and (2.18), for any entire $B_i(\lambda,t)$, equation (2.9) is equivalent to the following

$$\sum_{\rho=1}^{M}\sum_{i=1}^{N}\sum_{\alpha=0}^{n}\langle(H_{(A)i}P'^{(\alpha)}-P^{(\alpha)}H_{(A)i}),\,B_i(\Lambda_{(\rho)},t)(\Lambda_{(\rho)})^{\alpha}\overset{++}{\phi}{}^{F(A)}_{(\rho)F(A)}\rangle=0. \tag{2.19}$$

Further, equation (2.19) is equivalent to

$$\sum_{\rho=1}^{M}\left\langle\sum_{i=1}^{N}\sum_{\alpha=0}^{n}(\Lambda^{+}_{(\rho)})^{\alpha}B_i(\Lambda^{+}_{(\rho)},t)(H_{(A)i}P'^{(\alpha)}-P^{(\alpha)}H_{(A)i}),\,\overset{++}{\phi}{}^{F(A)}_{(\rho)F(A)}\right\rangle=0 \tag{2.20}$$

where $\Lambda^{+}_{(\rho)}$ are the operators adjoint to $\Lambda_{(\rho)}$ with respect to the bilinear form $\langle\chi,\psi\rangle_{F(A)}\overset{\mathrm{df}}{=}\langle\chi_{F(A)},\psi_{F(A)}\rangle$.

The equality (2.20) holds if the following system of equations is satisfied

$$\sum_{\alpha=0}^{n}\sum_{i=1}^{N}(\Lambda^{+}_{(\rho)})^{\alpha}B_i(\Lambda^{+}_{(\rho)},t)(H_{(A)i}P'^{(\alpha)}-P^{(\alpha)}H_{(A)i})=0 \qquad (\rho=1,\ldots,M). \tag{2.21}$$

Thus we find that the transformation $P^{(\alpha)}\to P'^{(\alpha)}$ which corresponds to the transformation $S\to S'$ of type (2.3) are of the form (2.21) where $B_i(\lambda,t)$ are arbitrary functions entire on λ. Let us point out that if there exists some relationship between $\Lambda^{+}_{(\rho)}$ then the functions $B_i(\lambda,t)$ $(i=1,\ldots,N)$ must satisfy the certain conditions in order that there may be no inconsistency in the system of equations (2.21) $(\rho=1,\ldots,M)$.

We see that the reason why we are restricted by the transformations $S \to S'$ of type (2.3) is because it is possible to convert these transformations into the form containing $P^{(\alpha)}$ and $P'^{(\alpha)}$ only (transformations (2.21)).

The transformation law (2.3) of the transition matrix S leads to a simple transformation law of the matrix of the scattering data $R(\lambda, t) \stackrel{\mathrm{df}}{=} S_{F(A)}(\lambda, t)(S_{0(A)}(\lambda))^{-1}$. It is easy to see that under transformation (2.3)

$$R(\lambda, t) \to R'(\lambda, t) = B^{-1}(\lambda, t) R(\lambda, t) B(\lambda, t).$$

3. General form of the integrable equations, Bäcklund transformations and integrals of motion

3.1. *The integrable equations*

The infinite-dimensional group of the transformations (2.21), (2.3) is a basis for the analysis of the nonlinear equations connected with (1.1) and their properties.

This group contains the transformations of different types. Let us consider the one-parameter subgroup of transformations given by the matrix

$$B = \sum_{i=1}^{N} \exp\left(-\mathrm{i} \int_t^{t'} \mathrm{d}s\, \Omega_i(\lambda, s)\right) \cdot H_{(A)i} \tag{3.1}$$

and $C = B$. It is not difficult to show that this transformation is a displacement in time t:

$$\begin{aligned} &S(\lambda, t) \to S'(\lambda, t) = \exp\left(\mathrm{i} \int_t^{t'} \sum_{i=1}^{N} \Omega_i(\lambda, s)\, \mathrm{d}s\, H_{(A)i}\right). \\ &S(\lambda, t) \exp\left(-\mathrm{i} \int_t^{t'} \mathrm{d}s \sum_{i=1}^{N} \Omega_i(\lambda, s) H_{(A)i}\right) = S(\lambda, t'). \end{aligned} \tag{3.2}$$

Correspondingly in the terms of $P^{(\alpha)}(x, t)$ this transformation $P^{(\alpha)}(x, t) \to P^{(\alpha)}(x, t')$ is of the form

$$\begin{aligned} \sum_{\alpha=0}^{n} \sum_{i=1}^{N} (\Lambda_{(\rho)}^{+})^{\alpha} \exp\left(-\mathrm{i} \int_t^{t'} \mathrm{d}s\, \Omega_i(\Lambda_{(\rho)}^{+}, s)\right)&(H_{(A)i} P^{(\alpha)}(x, t') \\ &- P^{(\alpha)}(x, t) H_{(A)i}) = 0 \qquad (\rho = 1, \ldots, M) \end{aligned} \tag{3.3}$$

where in the operator $\Lambda_{(\rho)}^{+}$ one must put $P'^{(\alpha)}(x, t) = P^{(\alpha)}(x, s)$. At $N = 2$, $n = 1$ this type of relationship was found by Calogero (1976). The relation (3.3) defines the evolution in time of the potentials $P^{(\alpha)}$: $P^{(\alpha)}(x, t) \to P^{(\alpha)}(x, t')$. The different evolution laws correspond to the different functions $\Omega_i(\lambda, t)$ $(i = 1, \ldots, N)$.

Let us consider the infinitesimal displacement $t \to t' = t + \varepsilon$, $\varepsilon \to 0$. In this case

$$\begin{aligned} &P(x, t') = P(x, t) + \varepsilon(\partial P/\partial t), \\ &B_i(\lambda, s) = 1 - \mathrm{i}\varepsilon \Omega_i(\lambda, t). \end{aligned} \tag{3.4}$$

Substituting (3.4) into (3.3) and taking into account only terms of the first order in ε we obtain

$$\sum_{\alpha=0}^{n} (L_{(\rho)}^{+})^{\alpha} \left(\frac{\partial P^{(\alpha)}}{\partial t} - \mathrm{i} \sum_{i=1}^{N} \Omega_i(L_{(\rho)}^{+}, t)[H_{(A)i}, P^{(\alpha)}]\right) = 0 \qquad (\rho = 1, \ldots, M) \tag{3.5}$$

where $L^+_{(\rho)} \overset{\mathrm{df}}{=} \Lambda^+_{(\rho)}(P'=P)$. Correspondingly for the transition matrix:

$$\frac{\mathrm{d}S(\lambda, t)}{\mathrm{d}t} = \mathrm{i}\left[\sum_{i=1}^{N} \Omega_i(\lambda, t) H_{(A)i}, S(\lambda, t)\right]. \tag{3.6}$$

Thus as the infinitesimal form of the transformations (3.3) we obtain the system of partial differential (in general integro-differential) equations (3.5). The system of relations (3.3) which does not contain the derivatives $\partial P^{(\alpha)}/\partial t$ is the 'integrated' form of the system of differential equations (3.5) and solves, but inexplicitly, the Cauchy problem for it.

The partial differential equations (3.5) are just the equations integrable by the IST method with the help of the polynomial spectral problem (1.1). For the applicability of the IST method in this case see Zakharov (1980).

A broader class (than (3.5)) of the integrable equations appears if $P^{(\alpha)}$ (as in the case $N=2$, $n=1$ considered by Calogero and Degasperis (1976)) depends on a few variables $t_1, \ldots, t_m$ of time type. Examining the $t_1, \ldots, t_m$ infinitesimal displacements we obtain from (3.3)

$$\sum_{\alpha=0}^{n} \left((L^+_{(\rho)})^{\alpha}\left\{\sum_{e=1}^{m} f_e(L^+_{(\rho)}, t_1, \ldots, t_m)\frac{\partial P^{(\alpha)}(x, t_1, \ldots, t_m)}{\partial t_e} - \mathrm{i}\sum_{i=1}^{M} \Omega_i(L^+_{(\rho)}, t_1, \ldots, t_m)[H_{(A)i}, P^{(\alpha)}]\right\}\right) = 0 \qquad (\rho = 1, \ldots, M) \tag{3.7}$$

where $f_e(\lambda, t_1, \ldots, t_m)$ $(e=1, \ldots, m)$ and $\Omega_i(\lambda, t_1, \ldots, t_m)$ $(i=1, \ldots, N)$ are arbitrary functions entire on λ.

3.2. The integrals of motion

We now point out that by virtue of (3.6) $S_{0(A)}$ is independent of time: $\mathrm{d}S_{0(A)}/\mathrm{d}t = 0$. (We consider the case of two independent variables x and t.) Hence $S_{0(A)}(\lambda)$ is a generating functional of the integrals of motion for equation (3.5). Expanding $\ln S_{0(A)}(\lambda)$ in a series on λ^{-1} in the usual way

$$\ln S_{0(A)}(\lambda) = \sum_{e=0}^{\infty} \lambda^{-e} C^{(e)} \tag{3.8}$$

we obtain the infinite series of the integrals of motion

$$\left\{C_i^{(e)}, \text{ where } C^{(e)} = \sum_{i=1}^{N} C_i^{(e)} H_{(A)i}, e = 1, 2, \ldots, \right\}.$$

The explicit expressions for $C^{(e)}$ in terms of $P^{\alpha}(x, t)$ can be found by standard procedure (for $n=1$ see e.g. Zakharov *et al* (1980)). Let us represent the fundamental matrix solution $F^+(x, t, \lambda)$ as follows:

$$F^+(x, t, \lambda) = R(x, t, \lambda)E(x, \lambda)\exp\left(\int_x^{\infty} \mathrm{d}y\, \chi(y, t, \lambda)\right) \tag{3.9}$$

where $E = \exp(\Sigma_{\alpha=0}^{n} \lambda^{\alpha} A^{(\alpha)} x)$, $\chi_{0(A)} = \chi$ and the matrix R satisfies the condition

$R_{0(A)}=1$. From (3.9) we have

$$\ln S_{0(A)}(\lambda)=\int_{-\infty}^{+\infty} \mathrm{d}y\, \chi(y,t,\lambda). \tag{3.10}$$

To obtain (3.10) the subalgebra $g_{0(A)}$ must be an Abelian.

Substituting (3.9) into (1.1) we obtain

$$\frac{\partial R}{\partial x}-\sum_{\alpha=0}^{n}\lambda^{\alpha}[A^{(\alpha)},R]-R\chi-\sum_{\alpha=0}^{n}\lambda^{\alpha}P^{(\alpha)}_{F(A)}R=0. \tag{3.11}$$

Let us expand χ and R in the asymptotic series in λ^{-1}

$$\begin{aligned}\chi(x,t,\lambda)&=\sum_{e=0}^{\infty}\lambda^{-e}\chi^{(e)}(x,t),\\ R(x,t,\lambda)&=1+\sum_{e=1}^{\infty}\lambda^{-e}R^{(e)}(x,t).\end{aligned} \tag{3.12}$$

Substituting these expansions into (3.11) and taking $0(A)$ and $F(A)$ projections of the relation obtained we get

$$\chi^{(e)}=-\sum_{\alpha=0}^{n}(P^{(\alpha)}_{F(A)}(x,t)R^{(e+\alpha)}(x,t))_{0(A)} \qquad e=1,2,\ldots \tag{3.13}$$

where the $R^{(e)}$ are determined by the following recursion relations

$$\begin{aligned}&P^{(n)}_{F(A)}=0,\\ &[A^{(n)},R^{(1)}]=-P^{(n-1)}_{F(A)},\\ &\quad\vdots\\ &[A^{(n)},R^{(e)}]+\sum_{\alpha=0}^{n-1}[A^{(\alpha)},R^{(e-\alpha)}]+P^{(e)}_{F(A)}\\ &\qquad+\sum_{\alpha=0}^{n-1}(P^{(\alpha)}_{F(A)}R^{(\alpha-e)})_{F(A)}=0, \qquad (e=n,n-1,\ldots,1,0)\\ &\frac{\partial R^{(e)}}{\partial x}-\sum_{\alpha=0}^{n}[A^{(\alpha)},R^{(e+\alpha)}]+\sum_{m=1}^{e}R^{(m)}\left(\sum_{\alpha=1}^{n-1}P^{(\alpha)}_{F(A)}R^{(e-m)}\right)_{0(A)}\\ &\qquad-\sum_{\alpha=0}^{n}(P^{(\alpha)}_{F(A)}R^{(e+\alpha)})_{F(A)}=0 \qquad (e=1,2,\ldots,).\end{aligned} \tag{3.14}$$

Formula (3.13) and (3.14) enable us to calculate all integrals of motion $C^{(e)}$ by recursion since due to (3.8) (3.10) and (3.12)

$$C^{(e)}=\int_{-\infty}^{+\infty}\mathrm{d}x\,\chi^{(e)}(x,t) \qquad (e=1,2,\ldots,). \tag{3.15}$$

Let us point out that for selfconsistency of the proposed procedure it is necessary that $P^{(n)}_{F(A)}=0$. Since $\det S=1$ we have $\operatorname{Tr} C^{(e)}=0$ $(l=1,2,\ldots,)$. So there are $N-1$ independent infinite series of the integrals of motion.

We emphasise that these integrals of motion $C^{(e)}$ $(e=1,2,\ldots,)$ are universal, i.e. they are integrals of motion for any equations of the type (3.5) (with any functions

$\Omega_i(\lambda, t)$). Indeed in the construction of the $C^{(e)}$ we use only the fact that $S_{0(A)}(\lambda)$ is independent of time and the spectral problem (1.1) but not of the form of the functions $\Omega_i(\lambda, t)$.

3.3. Bäcklund, and generalised Bäcklund transformations

The general structure of the transformations admissible by the integrable equations (3.5) are in essence the same as for $n=1$. (See Konopelchenko 1980a–d.) The group of transformations (2.21), (2.3) with $C=B$ is the universal symmetry group for the equation (3.5). Indeed these transformations change neither $S_{0(A)}(\lambda)$ nor, therefore, the Hamiltonians of equations (3.5). The integrals of motion $C^{(e)}$ are connected with these symmetry transformations. Further, the transformations (2.21) (2.3) with $C \neq B$ and B_i, C_i independent of time t are Bäcklund transformations (BT's) for the equations (3.5). The general structure of BT's are the same as in the case when $N=2$, $n=1$. (Konopelchenko 1979, 1980d): among them are discrete and continuous BT's. Then, transformations (2.21) with functions $B_i(\lambda, t)$ which depend on time t are generalised BT's. They convert the solutions of different equations of type (3.5) (with different functions $\Omega_i(\lambda, t)$) into each other. In particular, transformation (3.3) is the generalised BT from the equation $\Sigma_{\alpha=0}^{n}(L^{+}_{(p)})^{\alpha}(\partial P^{(\alpha)}/\partial t)=0$ to the equation (3.5) ($B_i = \exp(-i\int^{t}\mathrm{d}S\,\Omega_i(\lambda, s))$).

Thus, the universal group of the transformations (2.21) contains all the transformations characteristic of the integrable equations (3.5) (symmetry group, Bäcklund transformations) and these equations themselves.

4. First example: the linear bundle

Formulae (2.21) and (3.5) give us the general form of the transformations and integrable equations for the arbitrary polynomial bundle (1.1). The operators $\Lambda^{+}_{(p)}$ and $L^{+}_{(p)}$ play a central role in these formulae. In order to give an explicit description of the integrable equations and BT's we needed the explicit expressions for $\Lambda^{+}_{(p)}$ and $L^{+}_{(p)}$. The construction of the explicit expressions for $\Lambda^{+}_{(p)}$, $L^{+}_{(p)}$ is the main problem.

Here we consider two examples in which one can find operators $\Lambda^{+}(L^{+})$ explicitly.

For the linear bundle (Konopelchenko 1980a, b, c) N is arbitrary, $n=1$, $A^{(0)}=0$, $P^{(1)}_{F(A)}=0$, $A^{(1)}=\mathrm{i}A$, $P^{(0)}_{F(A)}=\mathrm{i}P$, i.e.

$$\partial\psi/\partial x = \mathrm{i}\lambda A\psi + \mathrm{i}P\psi. \tag{4.1}$$

In this case equation (2.15) is of the form $(\lambda - \Lambda_A)\overset{++}{\phi}{}^{F(A)}_{F(A)} = 0$ where Λ_A is an operator

$$\begin{aligned}\Lambda_A\phi \overset{\mathrm{df}}{=} ad_A^{-1}\Big\{&-\mathrm{i}\partial\phi/\partial x - (P'(x)\phi(x) - \phi(x)P(x))_{F(A)}\\ &-\mathrm{i}P'(x)\int_x^{\infty}\mathrm{d}y\,(P'(y)\phi(y) - \phi(y)P(y))_{0(A)}\\ &+\mathrm{i}\int_x^{\infty}\mathrm{d}y\,(P'(y)\phi(y) - \phi(y)P(y))_{0(A)}P(x)\Big\}\end{aligned} \tag{4.2}$$

where $[A, ad_A^{-1}\phi] \overset{\text{df}}{=} \phi$. The transformation (2.21) is the following

$$\sum_{i=1}^{N} B_i(\Lambda_A^+, t)(H_{(A)i}P' - PH_{(A)i}) = 0 \tag{4.3}$$

where $B_i(\lambda, t)$ are arbitrary entire functions and

$$\begin{aligned}\Lambda_A^+\phi = &-\mathrm{i}ad_A^{-1}(\partial\phi/\partial x) + (ad_A^{-1}\phi(x)P'(x) - P(x)ad_A^{-1}\phi(x))_{F(A)} \\ &-\mathrm{i}\int_{-\infty}^{x} \mathrm{d}y(ad_A^{-1}\phi(y)P'(y) - P(y)ad_A^{-1}\phi(y))_{0(A)}P'(x) \\ &+\mathrm{i}P(x)\int_{-\infty}^{x} \mathrm{d}y(ad_A^{-1}\phi(y)P'(y) - P(y)ad_A^{-1}\phi(y))_{0(A)}.\end{aligned} \tag{4.4}$$

The integrable equations are of the form

$$\frac{\partial P}{\partial t} - \mathrm{i}\sum_{i=1}^{N} \Omega_i(L_A^+, t)[H_{(A)i}, P]_{F(A)} = 0 \tag{4.5}$$

where

$$\begin{aligned}L_A^+ \overset{\text{df}}{=} \Lambda_A^+(P' = P) = &-\mathrm{i}ad_A^{-1}(\partial/\partial x) - [P(x), ad_A^{-1}\ \cdot]_{F(A)} \\ &-\mathrm{i}\left[P(x), \int_{-\infty}^{x} \mathrm{d}y[P(y), ad_A^{-1}\ \cdot]_{0(A)}\right].\end{aligned} \tag{4.6}$$

Results analogous to (4.2)–(4.6) have been obtained also in the cases where A is an arbitrary semi-simple matrix (Konopelchenko 1980b, c) and where elements of $P(x, t)$ belong to the infinite-dimensional abelian Z_2 graded algebra, i.e. among elements of $P(x, t)$ there are the usual (commutative) functions and anticommutative (Grassmann) variables (classical fermion field, see Konopelchenko (1980e)).

5. Second example: quadratic bundle of arbitrary order

We consider the quadratic ($n = 2$) bundle (1.1) with arbitrary N, $A^{(2)} = \mathrm{i}\alpha A$, $A^{(1)} = 2\mathrm{i}\beta A$, $A^{(0)} = 0$, $P_{F(A)}^{(2)} = 0$, $P_{F(A)}^{(1)} = \mathrm{i}\alpha P$, $P_{F(A)}^{(0)} = \mathrm{i}\beta P$ where α and β are arbitrary numbers, i.e.

$$\partial\psi/\partial x = \mathrm{i}(\alpha\lambda^2 + 2\beta\lambda)A\psi + \mathrm{i}(\alpha\lambda + \beta)P(x, t)\psi. \tag{5.1}$$

Let us consider two cases:

(a)
$$A = \begin{pmatrix} I_N & 0 \\ 0 & -1 \end{pmatrix}, \qquad P = \left(\begin{array}{c|c} \begin{matrix} & & \\ & 0_N & \\ & & \end{matrix} & \begin{matrix} q_1 \\ \vdots \\ q_N \end{matrix} \\ \hline r_1 \ldots r_N & 0 \end{array}\right) \tag{5.2}$$

where I_N is the identity matrix of the order N. The subalgebra $g_{0(A)}(=\{g, g \in gl(N, C), [g, A] = 0)$ consists of the matrices of the order $N+1$ of the type $\begin{pmatrix} \tilde{p} & 0 \\ 0 & b \end{pmatrix}$, where $\tilde{p}$ is an arbitrary $N \times N$ matrix and b is an arbitrary number.

(b) $$A=\begin{pmatrix} I_N & 0 \\ 0 & -I_N \end{pmatrix}, \qquad P=\begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix} \tag{5.3}$$

where Q and R are $N\times N$ matrices. In this case the subalgebra $g_{0(A)}$ consists of matrices of the type $\begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix}$ where p_1 and p_2 are arbitrary $N\times N$ matrices.

For $N=1$ (5.2) coincides with (5.3) and the corresponding spectral problem (5.1) coincides with those considered by Wadati *et al* (1979) (for $\beta=0$ see Kaup and Newell (1978)).

Let us now describe explicitly both the general form of the equations integrable by (5.1) and their Bäcklund transformations.

For arbitrary N in both of the cases (5.2) and (5.3) $[P(x), \phi_{F(A)}]_{F(A)}=0$. As a result equation (2.13) is of the form (we denote $\chi \overset{\text{df}}{=} \overset{++}{\phi}{}^{F(A)}_{F(A)}$):

$$\frac{\partial\chi}{\partial x}=\mathrm{i}(\alpha\lambda^2+2\beta\lambda)[A,\chi(x)]+(\alpha\lambda+\beta)^2P'(x)\int_x^\infty \mathrm{d}y(P'(y)\chi(y)-\chi(y)P(y))_{0(A)}$$
$$-(\alpha\lambda+\beta)^2\int_x^\infty \mathrm{d}y(P'(y)\chi(y)-\chi(y)P(y))_{0(A)}P(x). \tag{5.4}$$

One can rewrite equation (5.4) as follows $(ad_A^{-1}\chi=\frac{1}{2}A\chi)$

$$-\tfrac{1}{2}\mathrm{i}A(\partial\chi/\partial x)+\mathrm{i}\beta^2J\chi=(\alpha\lambda^2+2\beta\lambda)(1-\mathrm{i}\alpha J)\chi \tag{5.5}$$

where

$$J\chi \overset{\text{df}}{=} \tfrac{1}{2}AP'(x)\int_x^\infty \mathrm{d}y(P'(y)\chi(y)-\chi(y)P(y))_{0(A)}$$
$$-\tfrac{1}{2}A\int_x^\infty \mathrm{d}y(P'(y)\chi(y)-\chi(y)P(y))_{0(A)}P(x). \tag{5.6}$$

Therefore

$$\Lambda\chi=(\alpha\lambda^2+2\beta\lambda)\chi \tag{5.7}$$

where

$$\Lambda=(1-\mathrm{i}\alpha J)^{-1}[-\tfrac{1}{2}\mathrm{i}A(\partial/\partial x)+\mathrm{i}\beta^2J]$$
$$=\sum_{e=0}^{\infty}(\mathrm{i}\alpha)^eJ^e\left[-\frac{\mathrm{i}}{2}\frac{\partial}{\partial x}+\mathrm{i}\beta^2J\right]. \tag{5.8}$$

So, for the quadratic polynomial (2.14) we have $\bar{f}(\lambda)=\alpha\lambda^2+2\beta\lambda-\Lambda$. The 'roots' $\Lambda_{(1)}$ and $\Lambda_{(2)}$ are easily found

$$\Lambda_{(1)}=-\frac{\beta}{\alpha}+\left(\frac{\beta^2}{\alpha^2}+\frac{1}{\alpha^2}\Lambda^2\right)^{1/2},$$
$$\Lambda_{(2)}=-\frac{\beta}{\alpha}-\left(\frac{\beta^2}{\alpha^2}+\frac{1}{\alpha^2}\Lambda^2\right)^{1/2} \tag{5.9}$$

where the operator $Z^{1/2}$ is defined as $Z^{1/2}Z^{1/2}=Z$.

Similarly

$$\Lambda^{+}_{(1)}=-\frac{\beta}{\alpha}+\left(\frac{\beta^2}{\alpha^2}+\frac{1}{\alpha^2}(\Lambda^{+})^2\right)^{1/2},\qquad \Lambda^{+}_{(2)}=-\frac{\beta}{\alpha}-\left(\frac{\beta^2}{\alpha^2}+\frac{1}{\alpha^2}(\Lambda^{+})^2\right)^{1/2} \tag{5.10}$$

where

$$\Lambda^{+}=\left(-\frac{\mathrm{i}}{2}A\frac{\partial}{\partial x}+\mathrm{i}\beta^2J^{+}\right)(1-\mathrm{i}\alpha J^{+})^{-1} =\left(-\frac{\mathrm{i}}{2}A\frac{\partial}{\partial x}+\mathrm{i}\beta^2J^{+}\right)\sum_{e=0}^{\infty}(\mathrm{i}\alpha)^e(J^{+})^e \tag{5.11}$$

and

$$J^{+}\chi=-\frac{1}{2}P(x)\int_{-\infty}^{x}\mathrm{d}y(P(y)A\chi(y)-A\chi(y)P'(y))_{0(A)} +\frac{1}{2}\int_{-\infty}^{x}\mathrm{d}y(P(y)A\chi(y)-A\chi(y)P'(y))_{0(A)}P'(x). \tag{5.12}$$

The transformations (2.21) are the following

$$(\alpha\Lambda^{+}_{(1)}+\beta)\sum_{i=1}^{N^2+1(2N^2)}B_i(\Lambda^{+}_{(1)},t)(H_{(A)i}P'-PH_{(A)i})=0,\qquad (\alpha\Lambda^{+}_{(2)}+\beta)\sum_{i=1}^{N^2+1(2N^2)}B_i(\Lambda^{+}_{(2)},t)(H_{(A)i}P'-PH_{(A)i})=0 \tag{5.13}$$

where the matrices $H_{(A)i}$ ($i=1,\ldots,N^2+1$ for case (5.2); $i=1,\ldots,2N^2$ for case (5.3)) form a basis for the subalgebra $g_{0(A)}$. Since $\Lambda^{+}_{(2)}=-\Lambda^{+}_{(1)}-2\beta/\alpha$ the system of equation (5.13) is consistent if the functions $B_i(\lambda,t)$ satisfy the conditions $B_i(-\lambda-2\beta/\alpha,t)=B_i(\lambda,t)$ ($i=1,2,\ldots$). As a result $B_i(\lambda,t)=\tilde{B}_i(\alpha\lambda^2+2\beta\lambda,t)$ where $\tilde{B}_i(\mu,t)$ are arbitrary functions entire on $\mu=\alpha\lambda^2+2\beta\lambda$. By virtue of $\alpha(\Lambda^{+}_{(1)})^2+2\beta\Lambda^{+}_{(1)}=\Lambda^{+}$, (5.13) in this case is equivalent to the equation

$$\sum_{i=1}^{N^2+1(2N^2)}\tilde{B}_i(\Lambda^{+},t)(H_{(A)i}P'-PH_{(A)i})=0 \tag{5.14}$$

where the operator Λ^{+} is given by (5.11).

Correspondingly, the system of the equations (3.5) is consistent unchanged if $\Omega_i(-\lambda-(2\beta/\alpha),t)=\Omega_i(\lambda,t)$ (i.e. $\Omega_i(\lambda,t)=\tilde{\Omega}_i(\alpha\lambda^2+2\beta\lambda,t)$) and it is equivalent to the equation

$$\frac{\partial P}{\partial t}-\mathrm{i}\sum_{i=1}^{N^2+1(2N^2)}\tilde{\Omega}_i(L^{+},t)[H_{(A)i},P]=0 \tag{5.15}$$

where $L^{+}\overset{\mathrm{df}}{=}\Lambda^{+}(P'=P)=\alpha(L^{+}_{(1)})^2+2\beta L^{+}_{(1)}$, i.e.

$$L^{+}=\left(-\frac{\mathrm{i}}{2}A\frac{\partial}{\partial x}+\mathrm{i}\beta^2I^{+}\right)(1-\mathrm{i}\alpha I^{+})^{-1} \tag{5.16}$$

where

$$I^{-}\chi = -\frac{1}{2}\left[P(x), \int_{-\infty}^{x} \mathrm{d}y[P(y), A\chi(y)]_{0(A)}\right]. \tag{5.17}$$

It is easy to see that at $\alpha = 0$, $\beta = 1$ the transformations (5.14), equations (5.15) and operators Λ^{+}, L^{+} coincide with those for the linear bundle (§ 4).

The class of equations (5.15) contains the subclass of equations for which $\Sigma_{i=1} \tilde{\Omega}_i(\lambda, t)H_{(A)i} = \Omega(\lambda, t)A$ where $\Omega(\lambda, t)$ is an arbitrary function meromorphic on λ. These equations are of the form

$$\partial P/\partial t - 2\mathrm{i}\Omega(L^{+}, t)AP = 0. \tag{5.18}$$

For equations (5.15) and (5.18) and transformations (5.14) all the general properties considered in § 3 are of course valid.

Let us consider some particular cases of equations (5.18).

At $N = 1$ we have $(I^{+})^2 = 0$ and therefore

$$L^{+} = -\frac{\mathrm{i}}{2}A\frac{\partial}{\partial x} + \mathrm{i}\beta^2 I^{+} + \frac{\alpha}{2}A\frac{\partial}{\partial x}I^{+}. \tag{5.19}$$

Under an additional restriction $\beta = 0$, equations (5.18) for $N = 1$ are equivalent to the equations considered by Gerdjikov *et al* (1980)†. Among these equations there is the derivative nonlinear Schrödinger (DNLS) equation ($\Omega = -2\lambda^2$, $r = q^*$, α is an arbitrary real number):

$$\mathrm{i}\frac{\partial q}{\partial t} + \frac{\partial^2 q}{\partial x^2} - \mathrm{i}\alpha\frac{\partial}{\partial x}(|q|^2 q) = 0 \tag{5.20}$$

first considered by Kaup and Newell (1978).

For arbitrary β and $N = 1$ we have a class of equations (5.18) containing, in particular, the combined NLS equation ($\Omega = -2\lambda^2$, β is real):

$$\mathrm{i}\frac{\partial q}{\partial t} + \frac{\partial^2 q}{\partial x^2} + 2\beta^2|q|^2 q - \mathrm{i}\alpha\frac{\partial}{\partial x}(|q|^2 q) = 0 \tag{5.21}$$

which was introduced by Wadati *et al* (1979).

For arbitrary N, α and β and under the reduction (5.2), equations (5.18) are multicomponent generalisations of the equations for $N = 1$‡. In particular for $\Omega = -2\lambda^2$ equation (5.18) is

$$\begin{aligned} &\mathrm{i}\frac{\partial q_\delta}{\partial t} + \frac{\partial^2 q_\delta}{\partial x^2} + 2\beta^2 q_\delta \sum_{\gamma=1}^{N} q_\gamma r_\gamma - \mathrm{i}\alpha\frac{\partial}{\partial x}\left(q_\delta \sum_{\gamma=1}^{N} q_\gamma r_\gamma\right) = 0, \\ &\mathrm{i}\frac{\partial r_\delta}{\partial t} - \frac{\partial^2 r_\delta}{\partial x^2} - 2\beta^2 r_\delta \sum_{\gamma=1}^{N} q_\gamma r_\gamma - \mathrm{i}\alpha\frac{\partial}{\partial x}\left(r_\delta \sum_{\gamma=1}^{N} q_\gamma r_\gamma\right) = 0 \qquad (\delta = 1, \ldots, N). \end{aligned} \tag{5.22}$$

Under the reduction $r_\delta = q_\delta^*$ (α and β are real numbers) we obtain the multicomponent, combined NLS equation

$$\mathrm{i}\frac{\partial q_\delta}{\partial t} + \frac{\partial^2 q_\delta}{\partial x^2} + 2\beta^2 q_\delta \sum_{\gamma=1}^{N} (q_\gamma^* q_\gamma) - \mathrm{i}\alpha\frac{\partial}{\partial x}\left(q_\delta \sum_{\gamma=1}^{N} q_\gamma^* q_\gamma\right) = 0 \qquad (\delta = 1, \ldots, N) \tag{5.23}$$

† Let us point out that (5.18) (for $N = 1$, $\beta = 0$) is a more convenient and simpler form of the equations described by Gerdjikov *et al* (1980).

‡ For explicit calculations, the relations $I^{+}AP = 0$, $(I^{+})^2(\partial P/\partial x) = 0$ are useful.

which possesses SU(N) symmetry. At $N=2$ and $\beta=0$ equation (5.23) has been considered by Morris and Dodd (1979).

For the reduction (5.3) and arbitrary N the equations (5.18) are matrix generalisations of the equations with $N=1$. In particular at $\Omega=-2\lambda^2$ we have the system of matrix equations

$$\begin{aligned} &\mathrm{i}\frac{\partial Q}{\partial t}+\frac{\partial^2 Q}{\partial x^2}+\left(2\beta^2-\mathrm{i}\alpha\frac{\partial}{\partial x}\right)QRQ=0, \\ &\mathrm{i}\frac{\partial R}{\partial t}-\frac{\partial^2 R}{\partial x^2}-\left(2\beta^2+\mathrm{i}\alpha\frac{\partial}{\partial x}\right)RQR=0, \end{aligned} \tag{5.24}$$

where Q and R are $N\times N$ matrices. For real α and β, equations (5.24) permit the reduction $R=Q^+$, where + denotes Hermitian conjugation, and as a result we obtain the matrix combined NLS equation

$$\mathrm{i}\frac{\partial Q}{\partial t}+\frac{\partial^2 Q}{\partial x^2}+2\beta^2 QQ^+Q-\mathrm{i}\alpha\frac{\partial}{\partial x}(QQ^+Q)=0 \tag{5.25}$$

For $\Omega(\lambda)=4\lambda^3$ and arbitrary N we have the multicomponent and matrix combined modified Korteweg–de Vries equations.

All the equations (5.18) (and in particular equations (5.21)–(5.25)) are Hamiltonian ones and possess an infinite set of Hamiltonian structures. Namely, one can show that the equations (5.18) can be represented in the Hamiltonian form $\partial P/\partial t=\{P,\mathscr{H}_n\}_n$ with Poisson brackets

$$\{\bar{f},\mathscr{H}\}_n=\left\langle\frac{\delta\bar{f}}{\delta P^T},(L^+)^n\mathscr{D}\frac{\delta\mathscr{H}}{\delta P^T}\right\rangle \tag{5.26}$$

where $D=\partial/\partial x+\beta^2[P(x),\int_{-\infty}^x \mathrm{d}y[P(y),\cdot]_{0(A)}]$, the operator L^+ is given by formula (5.16) and n is any whole number. Equations (5.18) and their properties will be considered in further detail in a separate paper.

In the same manner one can also consider the quadratic bundle (5.1) under a more general reduction than (5.2) or (5.3), namely for

$$A=\begin{pmatrix} I_N & 0 \\ 0 & -I_M\end{pmatrix}, \qquad P=\begin{pmatrix} 0_N & Q \\ R & 0_M\end{pmatrix} \tag{5.27}$$

where $I_N(I_M)$ is the identical quadratic matrix of order $N(M)$, $0_N(0_M)$ is the zero quadratic matrix of order $N(M)$. The matrices Q and R are correspondingly $N\times M$ and $M\times N$ rectangular matrices. N and M are arbitrary numbers. In the particular cases $M=1$ and $M=N$ we have the corresponding reductions (5.2) and (5.3). In the case (5.27) the subalgebra $g_{0(A)}$ consists of matrices of type $\begin{pmatrix} P_1 & 0 \\ 0 & P_2\end{pmatrix}$ where P_1 is an arbitrary quadratic matrix of order N and P_2 is an arbitrary quadratic matrix of order M. Since in this case $[P,\phi_{F(A)}]_{F(A)}=0$ then all the formulae (5.4)–(5.18) are valid for the reduction (5.27) too†. In particular for $\Omega=-2\lambda^2$, equation (5.18) gives the following

$$\begin{aligned} &\mathrm{i}\frac{\partial Q}{\partial t}+\frac{\partial^2 Q}{\partial x^2}+\left(2\beta^2-\mathrm{i}\alpha\frac{\partial}{\partial x}\right)QRQ=0, \\ &\mathrm{i}\frac{\partial R}{\partial t}-\frac{\partial^2 R}{\partial x^2}-\left(2\beta^2+\mathrm{i}\alpha\frac{\partial}{\partial x}\right)RQR=0 \end{aligned} \tag{5.28}$$

† For properties of the rectangular matrices see e.g. Gantmakher (1967).

where Q and R are correspondingly $N\times M$ and $M\times N$ rectangular matrices. For real α, β and $R=Q^{+}$ we have

$$\mathrm{i}\frac{\partial Q}{\partial t}+\frac{\partial^2 Q}{\partial x^2}+2\beta^2 QQ^{+}Q-\mathrm{i}\alpha\frac{\partial}{\partial x}(QQ^{+}Q)=0 \tag{5.29}$$

For $M=1$ and $M=N$ equations (5.28) and (5.29) reduce correspondingly to equations (5.22)–(5.23) and to equations (5.24)–(5.25).

All the equations (5.18) corresponding to the reduction (5.27) (for any N and M) are Hamiltonian ones and possess an infinite family of Poisson brackets (5.26).

6. Polynomial and rational bundles as degenerate linear bundles

It is not difficult to see that, introducing the quantities $\psi_\alpha \overset{\mathrm{df}}{=} \lambda^\alpha\psi$, the polynomial spectral problem (1.1) can be represented in the following form† (assuming $P^{(n)}_{F(A)}=0$)

$$\begin{pmatrix}1 & 0 & \dots & 0\\ 0 & 0 & \dots & 0\\ \dots & \dots & \dots & \dots\\ 0 & 0 & \dots & 0\end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix}\psi_0\\ \psi_1\\ \vdots\\ \psi_{N-1}\end{pmatrix}$$

$$=\lambda\begin{pmatrix}0 & 0 & \dots & & A^{(n)}\\ 1 & 0 & \dots & & 0\\ 0 & 1 & \dots & & 0\\ \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & \dots & 1 & 0\end{pmatrix}\begin{pmatrix}\psi_0\\ \psi_1\\ \vdots\\ \vdots\\ \psi_{N-1}\end{pmatrix}+\begin{pmatrix}P^{(0)} & P^{(1)} & \dots & P^{(N-1)}\\ 0 & -1 & \dots & 0\\ \dots & \dots & \dots & \dots\\ 0 & \dots & -1 & 0\\ 0 & \dots & 0 & -1\end{pmatrix}\begin{pmatrix}\psi_0\\ \psi_1\\ \vdots\\ \vdots\\ \psi_{N-1}\end{pmatrix}. \tag{6.1}$$

Therefore the polynomial bundle (1.1) is no more than the special reduction of the degenerate linear bundle

$$B\frac{\partial}{\partial x}\chi=\lambda A\chi+P\chi \tag{6.2}$$

where $\det B=0$.

It is easy to see that the arbitrary rational bundle (Zakharov 1980)

$$\frac{\partial\psi}{\partial x}=\left(\sum_{\alpha=0}^{N_1}\lambda^\alpha P^{(\alpha)}(x,t)+\sum_{\beta,\gamma=1}^{N_2}\frac{Q^{(\beta\gamma)}(x,t)}{(\lambda-\lambda_{0\beta})^\gamma}\right)\psi. \tag{6.3}$$

can also be represented in the form of (6.2).

The polynomial (6.1) and rational (6.3) bundles are irreducible forms of the degenerate linear bundles. In particular (1.1) is the irreducible form of (6.1). In the case $\det B\neq 0$ the spectral problem (6.2) is reduced to (4.1).

So the analysis of the polynomial and general rational bundles is equivalent to the analysis of the degenerate linear bundle (6.2) under the special reductions.

† The author is grateful to Dr P P Kulish for a stimulating discussion of this point.

Furthermore, the degenerate bundle (6.2) allows a natural multi-dimensional generalisation

$$\sum_{i=1}^{n} B_i \frac{\partial}{\partial x^i} \chi = \lambda A \chi + P(x_1, \ldots, x_n)\chi \tag{6.4}$$

where B_i are matrices and $x_1, \ldots, x_n$ are independent variables (Zakharov and Shabat 1979). For these reasons, the generalisation of the AKNS method to the degenerate linear bundle (6.2) would appear to be very useful.

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